*Int. J. Heat Mass Transfer.* Vol. 8, pp. 499–505. Pergamon Press 1965. Printed in Great Britain

# APPROXIMATE EIGENVALUES FOR HEAT TRANSFER TO LAMINAR OR TURBULENT FLOW IN AN ANNULUS

ALLYN J. ZIEGENHAGEN

Department of Physics, University of Wisconsin, Milwaukee

(*Received* 11 *June* 1963 *and in revised form* 25 *May* 1964)

**Abstract**—Approximate expressions for the higher eigenvalues for heat transfer to a fluid in turbulent or laminar Newtonian or non-Newtonian axial motion through an annulus are obtained by the WKB method. A comparison of these eigenvalues for a specific heat-transfer problem for laminar Newtonian flow with results from the literature is made. The higher eigenvalues agree quite well, but the lowest eigenvalue is as much as 40 per cent too high.

## NOMENCLATURE

$A$, constant of integration;
$B$, constant of integration;
$C_1$, $C_2$, constants of integration;
$C_i$, expansion coefficient;
$\hat{C}_p$, heat capacity at constant pressure, per unit of mass;
$f(\rho)$, dimensionless velocity profile in terms of $\rho$;
$g(\rho)$, dimensionless thermal conductivity in terms of $\rho$;
$h(\beta)$, dimensionless velocity profile in terms of $\beta$;
$h(0)$, $h'(0)$, dimensionless velocity and first derivative with respect to $\beta$ evaluated at $\beta = 0$;
$j(\beta)$, dimensionless thermal conductivity in terms of $\beta$;
$j(0)$, $j'(0)$, dimensionless thermal conductivity and first derivative with respect to $\beta$ evaluated at $\beta = 0$;
$J_{1/3}$, $J_{-1/3}$, Bessel functions;
$k(\gamma)$, dimensionless velocity profile in terms of $\gamma$;
$k$, $k^{(l)}$, overall thermal conductivity; thermal conductivity of stagnant fluid;
$k(0)$, $k'(0)$, dimensionless velocity and first derivative evaluated at $\gamma = 0$;
$L$, dimensionless axial length used by Hatton and Quarmby;
$m(\gamma)$, dimensionless thermal conductivity in terms of $\gamma$;
$n$, index;
$q_0$, uniform heat flux *into* fluid at outer annulus wall;
$r$, $R$, radial coordinate, outer radius of annulus;
$\bar{T}$, time-smooth temperature;
$\bar{T}_0$, uniform fluid temperature at $z = 0$;
$T_{iw}$, uniform temperature at the inner wall of the annulus;
$\langle \bar{T} \rangle$, bulk fluid temperature;
$\bar{V}_z(r)$, $\bar{V}_{z,\max}$, velocity profile, maximum velocity;
$z$, axial direction;
$\beta$, dimensionless distance from outer wall of annulus measured inwards;
$\Gamma(\gamma)$, eigenfunction;
$\gamma$, dimensionless distance from inner wall of annulus measured outwards;
$\zeta$, dimensionless axial distance;
$\Theta$, $\Theta_\infty$, dimensionless temperature; dimensionless temperature at very large distances from $z = 0$;
$\kappa$, ratio of radius of inner wall to that of outer wall;
$\lambda$, eigenvalue;
$\lambda_{HQ}$, eigenvalues given by Hatton and Quarmby;
$\lambda'_{HQ}$, eigenvalues evaluated from the WKB expression but recalculated for direct comparison with $\lambda_{HQ}$;
$\xi$, dummy variable of integration;
$\rho$, dimensionless radial distance, density;

$\phi$, eigenfunction;
$\psi$, a part of the dimensionless temperature;
$\Omega$, eigenfunction.

## INTRODUCTION

MANY problems of heat transfer to flowing fluids can be reduced to differential equations, the solution of which requires the determination of a set of quantities called eigenvalues. Approximate expressions for the higher valued eigenvalues can be obtained by the WKB method which was first applied to laminar flow heat transfer problems by Sellars, Tribus, and Klein [6] in 1956. Their solution for laminar flow in a round tube has been extended by Sternling and Sleicher [7] to heat transfer to a fluid in turbulent motion through a round tube. In 1958, Dzung [3] in his consideration of the heat transfer to a fluid in motion through a round tube with a sinusoidal heat flux distribution at the wall, obtained approximate expressions for the higher valued eigenvalues. He also considered, as a limiting case of the truncated half-wave, the uniform heat flux problem. Lundberg, McCuen, and Reynolds [5] have considered the general problem of heat transfer to a fluid in laminar Newtonian motion through an annulus. In this note, an expression for the higher eigenvalues will be obtained for one case of heat transfer to a fluid in turbulent or laminar Newtonian or non-Newtonian motion through an annulus. Eigenvalues for the laminar flow of a Newtonian fluid are calculated from the WKB expression and compared with the results of Hatton and Quarmby [4].

In particular, for $z < 0$, a fluid flowing in the $(+z)$ direction is considered to have a uniform temperature $\bar{T}_0$ and a fully developed velocity profile $\bar{V}_z(r)$. At $z = 0$, the fluid enters a heat transfer region where a uniform temperature, $T_{iw}$, is imposed on the inner wall ($r = \kappa R$) and a uniform heat flux, $q_0$, *into* the fluid at the outer. If it is assumed that steady state exists, that the time-smoothed velocity profile is fully developed, that the fluid is incompressible, and that viscous dissipation and longitudinal heat conduction are relatively unimportant, the equation of energy is reduced to (Bird *et al.* [21]),

$$\rho \hat{C}_p \bar{V}_z \frac{\partial \bar{T}}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial \bar{T}}{\partial r}\right). \qquad (1)$$

The thermal conductivity, $k$, is assumed to be a constant for laminar motion and, for turbulent motion, to be strongly dependent on the degree of turbulence. The quantities with bars over them, $\bar{V}_z$ and $\bar{T}$, are interpreted as time smoothed for turbulent motion.

Equation (1), when the substitutions:

$$g(\rho) = k/k^{(l)} \qquad (2a)$$

$$\rho = r/R \qquad (2b)$$

$$f(\rho) = \bar{V}_z/\bar{V}_{z,\max} \qquad (2c)$$

$$\zeta = z\bigg/\left[\left(\frac{\rho \hat{C}_p}{k^{(l)}}\right) R^2 \bar{V}_{z,\max}\right] \qquad (2d)$$

and

$$\Theta = (\bar{T} - T_{iw})/(\bar{T}_0 - T_{iw}) \qquad (2e)$$

are made, becomes

$$f(\rho)\frac{\partial \Theta}{\partial \zeta} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(g\rho\frac{\partial \Theta}{\partial \rho}\right). \qquad (3)$$

$k^{(l)}$ is the value of the thermal conductivity for laminar motion.

The dimensionless boundary conditions become

$$\rho = \kappa, \quad \Theta = 0 \qquad (4a)$$

$$\rho = 1, \quad \frac{\partial \Theta}{\partial \rho} = \frac{R q_0}{k^{(l)}(\bar{T}_0 - T_{iw})} \qquad (4b)$$

$$\zeta = 0, \quad \Theta = 1. \qquad (4c)$$

The problem is now converted to a homogeneous form by the substitution

$$\Theta(\rho, \zeta) = \Theta_\infty(\rho) + \Psi(\rho, \zeta) \qquad (5)$$

and the requirement that $\Theta_\infty(\rho)$ satisfy the nonhomogeneous part of the boundary conditions. $\Theta_\infty(\rho)$ is found to be:

$$\Theta_\infty(\rho) = \frac{R q_0}{k^{(l)}(\bar{T}_0 - T_{iw})} \cdot \int_\kappa^\rho \frac{d\xi}{\xi\, g(\xi)}. \qquad (6)$$

The differential equation for $\Psi(\rho, \zeta)$ becomes

$$f(\rho)\frac{\partial \Psi}{\partial \zeta} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(g\rho\frac{\partial \Psi}{\partial \rho}\right) \qquad (7)$$

and the homogeneous boundary conditions are:

$$\rho = \kappa, \quad \Psi = 0 \tag{8a}$$

$$\rho = 1, \quad \frac{\partial \Psi}{\partial \rho} = 0 \tag{8b}$$

$$\zeta = 0, \quad \Psi(\rho, 0) = 1 - \Theta_\infty(\rho). \tag{8c}$$

The solution to this problem is

$$\Psi(\rho, \zeta) = \sum_{i=0}^{\infty} C_i \exp\left[-\lambda_i^2 \zeta\right] \phi_i(\rho) \tag{9}$$

where

$$C_i = \frac{\int_\kappa^1 \rho f(\rho)\, \Psi(\rho, 0)\, \phi_i(\rho)\, d\rho}{\int_\kappa^1 \rho f(\rho)\, \phi_i^2(\rho)\, d\rho} \tag{10}$$

and the function $\phi_i(\rho)$ is the solution of

$$\frac{d}{d\rho}\left(g\, \rho \frac{d\phi_i}{d\rho}\right) + \lambda_i^2\, \rho f(\rho)\, \phi_i(\rho) = 0 \tag{11}$$

and

$$\phi_i(\kappa) = \phi_i'(1) = 0. \tag{12}$$

## EVALUATION OF THE EIGENVALUE

Approximate expressions for the higher values of the discrete eigenvalues $\lambda_i$ are herein obtained; the procedure of analysis is quite similar to that used by Sellars *et al.* [6].

In terms of the dimensionless distance from the outer wall of the annulus, $\beta = 1 - \rho$, the WKB expression for the eigenfunction $\phi(\rho) = \Omega(\beta)$ is

$$\Omega(\beta) = \frac{A \exp\left\{+(\sqrt{-1})\, \lambda \int_0^\beta \sqrt{[h(\xi)/j(\xi)]}\, d\xi\right\} + B \exp\left\{-(\sqrt{-1})\lambda \int_0^\beta \sqrt{[h(\xi)/j(\xi)]}\, d\xi\right\}}{(1-\beta)^{1/2}\, [h(\beta)\, j(\beta)]^{1/4}}. \tag{13}$$

The additional functional changes

$$g(\rho) = j(\beta); \quad f(\rho) = h(\beta) \tag{14a, b}$$

have been made.

The boundary conditions to be satisfied are,

$$\Omega(1 - \kappa) = \Omega'(0) = 0. \tag{15}$$

Equation (13) has singularities at $\beta = 0$ and $\beta = 1 - \kappa$ since, at these points, the dimensionless velocity $h(\beta)$ is zero. Hence, the constants $A$ and $B$ cannot be determined directly. Instead, solutions of approximate expressions of equation (11) will be obtained for regions close enough to either wall so that turbulence and the effects of the circular geometry may be neglected and so that the velocity profile may be assumed linear. Then, as indicated below, the constants $A$ and $B$ and the eigenvalue $\lambda$ will be obtained by matching the several approximate solutions and satisfying the boundary conditions.

When $\beta \cong 0$, equation (11) is approximated by

$$\frac{d^2\Omega}{d\beta^2} + \lambda^2\, h'(0)\, .\, \beta\, .\, \Omega = 0 \tag{16}$$

where the approximations,

$$f(\rho) = h(\beta) = h(0) + h'(0)\, .\, \beta + \ldots \cong h'(0)\, .\, \beta$$

$$\rho = 1 - \beta \cong 1$$

$$g(\rho) = j(\beta) = j(0) + j'(0)\, .\, \beta + \ldots \cong j(0) = 1$$

are made. The solution of equation (16) is

$$\Omega(\beta) = \beta^{1/2}\left[C_1 J_{1/3}\left(\frac{2\lambda}{3}\sqrt{[h'(0)]}\, \beta^{3/2}\right) + C_2 J_{-1/3}\left(\frac{2\lambda}{3}\sqrt{[h'(0)]}\, \beta^{3/2}\right)\right]. \tag{17}$$

To satisfy the condition $\Omega'(0) = 0$, $C_1$ is set equal to zero. When $C_2$ is arbitrarily set equal to unity,

$$\Omega(\beta) = \beta^{1/2} J_{-1/3}\left(\frac{2\lambda}{3}\sqrt{[h'(0)]}\, \beta^{3/2}\right). \tag{18}$$

Equations (13) and (18) are now matched. For very small values of $\beta$,

$$\int_0^\beta \sqrt{\left[\frac{h(\xi)}{j(\xi)}\right]}\, d\xi \cong \int_0^\beta \sqrt{[h'(0)]}\, \xi^{1/2}\, d\xi = \frac{2}{3}\sqrt{[h'(0)]}\, \beta^{3/2} \tag{19}$$

and for very large values of $\lambda$ (large enough so that $\lambda\beta^{3/2}$ is quite large),

$$J_{-1/3}\left(\frac{2\lambda}{3}\sqrt{[h'(0)]}\,\beta^{3/2}\right) \cong \left(\frac{3}{\pi\lambda\sqrt{[h'(0)]}}\right) \cdot \frac{\cos\{(2\lambda/3)\sqrt{[h'(0)]}\,\beta^{3/2} - (\pi/12)\}}{\beta^{3/4}}. \quad (20)$$

These approximations cause equations (13) and (18) to exactly match, and allow $A$ and $B$ to be determined. Equation (13) then becomes

$$\Omega(\beta) = \left(\frac{3}{\pi\lambda(1-\beta)}\right)^{1/2} \cdot \frac{\cos\{\lambda\int_0^\beta \sqrt{[h(\xi)/j(\xi)]}\,d\xi - (\pi/12)\}}{[j(\beta)\,.\,h(\beta)]^{1/4}}. \quad (21)$$

Very close to the inner wall of the annulus, equation (11) may be converted to the form,

$$\frac{d^2\Gamma}{d\gamma^2} + \lambda^2 k'(0)\,.\,\gamma\,.\,\Gamma = 0 \quad (22)$$

when the substitutions

$$\gamma = \rho - \kappa$$

$$f(\rho) = k(\gamma) = k(0) + k'(0).\gamma + \ldots \cong k'(0)\,.\,\gamma$$

$$g(\rho) = m(\gamma) \cong 1$$

$$\phi(\rho) = \Gamma(\gamma)$$

are made. The general solution of equation (22) is

$$\Gamma = \gamma^{1/2}\left[C_1 J_{1/3}\left(\frac{2\lambda}{3}\sqrt{[k'(0)]}\,\gamma^{3/2}\right) + C_2 J_{-1/3}\left(\frac{2\lambda}{3}\sqrt{[k'(0)]}\,\gamma^{3/2}\right)\right]. \quad (23)$$

The constants $C_1$ and $C_2$ are determined by matching equation (23), approximated for large values of $\lambda\gamma^{3/2}$ with equation (21); equation (23) becomes, after $C_1$ and $C_2$ are determined,

$$\Gamma = \frac{2\gamma^{1/2}}{(3\kappa)^{1/2}}\left\{\sin\left[\lambda\int_0^{1-\kappa}\sqrt{\left(\frac{h(\xi)}{j(\xi)}\right)}\,d\xi - \frac{\pi}{6}\right] J_{1/3}\left(\frac{2\lambda}{3}\sqrt{[k'(0)]}\,\gamma^{3/2}\right) - \sin\left[\lambda\int_0^{1-\kappa}\sqrt{\left(\frac{h(\xi)}{j(\xi)}\right)}\,d\xi - \frac{\pi}{2}\right] J_{-1/3}\left(\frac{2\lambda}{3}\sqrt{[k'(0)]}\,\gamma^{3/2}\right)\right\}. \quad (24)$$

To satisfy the condition $\Gamma(0) = 0$, it is necessary to set

$$\sin\left[\lambda\int_0^{1-\kappa}\sqrt{\left(\frac{h(\xi)}{j(\xi)}\right)}\,d\xi - \frac{\pi}{2}\right] = 0.$$

It follows that the eigenvalue can take on the values

$$\lambda_n = \frac{(n+\frac{1}{2})\pi}{\int_0^{1-\kappa}\sqrt{[h(\xi)/j(\xi)]}\,d\xi}; \quad n = 0, 1, 2, \ldots \quad (25)$$

## CONCLUSIONS

In the inlet section of annular heat transfer equipment, it is necessary to keep a relatively large number of terms in equation (9). In the author's opinion, one of the most difficult items to evaluate is the higher valued ($n = 3$, 4, etc.) eigenvalue. Equation (25) should provide a direct and quite accurate expression for the evaluation of these terms.

Eigenvalues have been calculated, with the aid of an IBM 1620 computer, from equation (25) for the laminar flow of a Newtonian fluid in an annulus with values of

$$\kappa = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{20}{21}.$$

These results are modified and tabulated in Table 1 and compared with those obtained by Hatton and Quarmby [4]. The eigenvalues cannot be compared directly since a different choice of the dimensionless axial distance was made in the two analyses. The comparison may be made after setting

$$(\lambda'_{HQ})^2 L = \lambda^2 \zeta \quad (26)$$

where $L$ and $\lambda'_{HQ}$ are the dimensionless axial distance and eigenvalue used by Hatton and Quarmby. The dimensionless laminar Newtonian velocity profile is, [2],

$$f(\rho) = \frac{V_z}{V_{z,\max}} = \frac{1 - \rho^2 + [(1 - \kappa)^2/\ln(1/\kappa)] \ln \rho}{1 - [(1 - \kappa^2)/2 \ln(1/\kappa)] \{1 - \ln[(1 - \kappa^2)/2 \ln(1/\kappa)]\}}, \tag{27}$$

and $h(\xi)$, obtained by replacing $\rho$ by $1 - \xi$, is

$$h(\xi) = \frac{2\xi - \xi^2 + [(1 - \kappa^2)/\ln(1/\kappa)] \ln(1 - \xi)}{1 - [(1 - \kappa^2)/2 \ln(1/\kappa)] \{1 - \ln[(1 - \kappa^2)/2 \ln(1/\kappa)]\}}. \tag{28}$$

After the necessary manipulations with equations (26), (27), and (28) have been carried out, the WKB eigenvalues recalculated for direct comparison with those of Hatton and Quarmby are

$$\lambda'_{HQ} = \left[\frac{1 + \kappa^2 - [(1 - \kappa^2)/\ln(1/\kappa)]}{2}\right]^{1/2} \cdot \frac{\pi(1 - \kappa)(n + \frac{1}{2})}{\int_0^{1-\kappa} \sqrt{\{2\xi - \xi^2 + [(1 - \kappa^2)/\ln(1/\kappa)] \ln(1 - \xi)\}}\, d\xi}. \tag{29}$$

The dimensionless thermal conductivity, $g(\xi) = 1$.

*Table 1. Comparison of eigenvalues calculated from WKB approach ($\lambda'_{HQ}$) and by Hatton and Quarmby ($\lambda_{HQ}$, reference [4])*

| $n$ | $\lambda'_{HQ}$ | $\lambda_{HQ}$ | $\lambda_{HQ}' - \lambda_{HQ}$ | $\frac{\lambda_{HQ}' - \lambda_{HQ}}{\lambda_{HQ}} . 100$ |
|---|---|---|---|---|
| | | $\kappa = 1/6$ | | |
| 0 | 1·59395 | 1·12495 | 0·46900 | 41·7 |
| 1 | 4·78186 | 4·58269 | 0·19917 | 4·35 |
| 2 | 7·96977 | 7·83073 | 0·13904 | 1·78 |
| 3 | 11·15768 | 11·04763 | 0·11005 | 1·00 |
| 4 | 14·34558 | 14·25304 | 0·09254 | 0·65 |
| 5 | 17·53349 | 17·45278 | 0·08071 | 0·46 |
| 6 | 20·72140 | 20·64929 | 0·07211 | 0·35 |
| 7 | 23·90930 | 23·84375 | 0·06555 | 0·27 |
| 8 | 27·09721 | 27·03685 | 0·06036 | 0·22 |
| 9 | 30·28512 | 30·22899 | 0·05613 | 0·19 |
| | | $\kappa = 1/3$ | | |
| 0 | 1·61644 | 1·28216 | 0·33428 | 26·1 |
| 1 | 4·84934 | 4·70436 | 0·14498 | 3·08 |
| 2 | 8·08222 | 7·98144 | 0·10078 | 1·26 |
| 3 | 11·31512 | 11·23569 | 0·07943 | 0·71 |
| 4 | 14·54800 | 14·48154 | 0·06646 | 0·46 |
| 5 | 17·78090 | 17·72327 | 0·05763 | 0·33 |
| 6 | 21·01378 | 20·96265 | 0·05113 | 0·24 |
| 7 | 24·24668 | 24·20056 | 0·04612 | 0·19 |
| 8 | 27·47956 | 27·43748 | 0·04208 | 0·15 |
| 9 | 30·71246 | 30·67373 | 0·03873 | 0·13 |
| | | $\kappa = 1/2$ | | |
| 0 | 1·62608 | 1·38300 | 0·24308 | 17·6 |
| 1 | 4·87823 | 4·76771 | 0·11052 | 2·32 |
| 2 | 8·13039 | 8·05276 | 0·07763 | 0·96 |
| 3 | 11·38255 | 11·32103 | 0·06152 | 0·54 |
| 4 | 14·63470 | 14·58304 | 0·05166 | 0·35 |
| 5 | 17·88686 | 17·84195 | 0·04491 | 0·25 |
| 6 | 21·13901 | 21·09910 | 0·03991 | 0·19 |
| 7 | 24·39117 | 24·35514 | 0·03603 | 0·15 |
| 8 | 27·64333 | 27·61044 | 0·03289 | 0·12 |
| 9 | 30·89548 | 30·86523 | 0·03025 | 0·10 |

*Table 1—continued*

| $n$ | $\lambda'_{HQ}$ | $\lambda_{HQ}$ | $\lambda'_{HQ} - \lambda_{HQ}$ | $\frac{\lambda'_{HQ} - \lambda_{HQ}}{\lambda_{HQ}} . 100$ |
|---|---|---|---|---|
| | | $\kappa = 2/3$ | | |
| 0 | 1·63055 | 1·45640 | 0·17415 | 12·0 |
| 1 | 4·89166 | 4·80780 | 0·08386 | 1·74 |
| 2 | 8·15276 | 8·09310 | 0·05966 | 0·74 |
| 3 | 11·41387 | 11·36629 | 0·04758 | 0·42 |
| 4 | 14·67498 | 14·63486 | 0·04012 | 0·27 |
| 5 | 17·93608 | 17·90112 | 0·03496 | 0·20 |
| 6 | 21·19719 | 21·16606 | 0·03113 | 0·15 |
| 7 | 24·45830 | 24·43017 | 0·02813 | 0·12 |
| 8 | 27·71940 | 27·69372 | 0·02568 | 0·09 |
| 9 | 30·98051 | 30·95689 | 0·02362 | 0·08 |
| | | $\kappa = 20/21$ | | |
| 0 | 1·63118 | 1·54688 | 0·08430 | 5·45 |
| 1 | 4·89354 | 4·85174 | 0·04180 | 0·86 |
| 2 | 8·15590 | 8·13040 | 0·02550 | 0·31 |
| 3 | 11·41826 | 11·40302 | 0·01524 | 0·13 |
| 4 | 14·68062 | 14·67321 | 0·00741 | 0·05 |
| 5 | 17·94298 | 17·94215 | 0·00083 | 0·005 |
| 6 | 21·20534 | 21·21035 | −0·00501 | −0·02 |
| 7 | 24·46770 | 24·47810 | −0·01040 | −0·04 |
| 8 | 27·73006 | 27·74554 | −0·01548 | −0·06 |
| 9 | 30·99242 | 31·01279 | −0·02037 | −0·07 |

It is evident from Table 1 that the WKB approach is an accurate means for computing eigenvalues beyond the third or fourth. The first two or three are quite inaccurate; the inaccuracy of the lower eigenvalues in problems of heat transfer to fluids in laminar motion in round tubes has been pointed out by Beek and Eggink [1]. This makes the evaluation of limiting Nusselt numbers that depend primarily on the lower eigenvalues quite inaccurate.

It is also noted that, for a given value of the index $n$, the agreement between the approximate and exact eigenvalues becomes better as $\kappa$ increases. This seems reasonable because curvature has been neglected in the approximate equation (22), which should be more accurate for a higher value of $\kappa$.


## ACKNOWLEDGEMENT

The author wishes to thank the Kimberley Clark Corporation which provided a graduate fellowship during the period in which the above work was completed at the University of Wisconsin, Madison.


## REFERENCES


1. W. J. Beek and R. Eggink, Heat transfer to a non-Newtonian fluid in laminar flow in a circular tube, *De Ingenieur* **74**, 81–89 (1962).
2. R. B. Bird, W. E. Stewart and E. N. Lightfoot, *Transport Phenomena*. Wiley, New York (1960).
3. L. S. Dzung, Heat transfer in round duct with sinusoidal heat flux distribution, *Proc. 2nd U.N. Int. Conf. Peaceful Uses, Atomic Energy* **7**, 657 (1958).
4. A. P. Hatton and A. Quarmby, Heat transfer in the thermal entry length with laminar flow in an annulus, *Int. J. Heat Mass Transfer* **5**, 973 (1962).
5. R. E. Lundberg, P. A. McCuen and W. C. Reynolds, Heat transfer in annular passages. Hydrodynamically developed laminar flow with arbitrarily prescribed wall temperatures or heat fluxes, *Int. J. Heat Mass Transfer* **6**, 495 (1963).
6. J. R. Sellars, M. Tribus and J. S. Klein, Heat transfer to laminar flow in a round tube or flat conduit—The Graetz Problem extended, *Trans. Amer. Soc. Mech. Engrs* **78**, 441 (1956).
7. C. V. Sternling and C. V. Sleicher, Asymptotic eigenfunction for turbulent heat transfer in a pipe, *J. Aerospace Sci.* **29**, 109 (1962).
8. I. R. Whiteman and W. B. Drake, Heat transfer to flow in a round tube with arbitrary velocity distribution, *Trans. Amer. Soc. Mech. Engrs* **80**, 728 (1958).

**Résumé**—Des expressions approchées pour les valeurs propres élevées pour le transport de chaleur à un fluide en mouvement axial Newtonien ou non-Newtonien, laminaire ou turbulent, à travers un tuyau annulaire sont obtenues par la méthode WKB. Une comparaison de ces valeurs propres pour un problème spécifique de transport de chaleur avec un écoulement Newtonien laminaire est faite avec les résultats de la bibliographie. Les valeurs propres élevées sont en bon accord, mais la valeur propre la plus basse est 40% plus élevée.

**Zusammenfassung**—Mit der WKB-Methode werden Näherungsausdrücke für die höheren Eigenwerte beim Wärmeübergang an eine Flüssigkeit von turbulenter oder laminarer Newtonscher oder nicht Newtonscher axialer Bewegung durche einen Ringraum erzielt. Für ein Einzelproblem des Wärmeüberganges bei laminarer Newtonscher Strömung werden diese Eigenwerte mit Ergebnissen aus der literatur verglichen. Die höheren Eigenwerte stimmen gut überein, aber der unterste Eigenwert liegt um 40% zu hoch.

**Аннотация**—Методом ВКБ получены приближенные выражения для высших собтвенных значений для переноса в круглом кольце к жидкости, находящейся в турбулентном или ламинарном ньютоновском или неньютоновском осесимметричном движении. Проведено сравнение этих собственных значений для частной задачи теплообмена при ламинарном ньютоновском течении с известными литературными данными. Для высших собственных значений согласие вполне хорошее, но низшие собственные значения приблизительно на 40 процентов превосходят значения, приведенные в литературе.